Pergamon

Int. J. Multiphase Flow Vol. 22, No. 5, pp. 1005–1021, 1996

PII: S0301-9322(96)00022-5
0301-9322/96 $15.00 + 0.00

DISPERSION OF INITIALLY-STATIC AND INITIALLY-EXCITED PARTICLES IN A TURBULENT FLUID FLOW

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(Received 13 August 1995; in revised form 14 April 1996)

Abstract—The motion of a particle released into a fluid flow depends upon the velocity of the fluid in the immediate vicinity of the particle. In a turbulent flow, the fluid velocity is a random quantity, so that particle motions are also random. Knowledge of certain statistical characteristics of the fluid motion allows for determination of mean properties of the particle motions. In this paper, dispersion of particles is analysed by assuming a simple form for the auto-correlation of the fluid motions following the particle in homogeneous, isotropic and stationary turbulence. Two cases are considered, one where the particles are initially at rest, the second where the particles are initially 'excited'. Particle velocity auto-correlations, particle dispersion coefficients and related quantities are determined as functions of time for both cases. Expressions for development times for heavy particles are derived. Copyright © 1996 Elsevier Science Ltd.

Key Words: particle dispersion, turbulent flows, gas/particle flows

1. INTRODUCTION

The dispersion of discrete particles under the influence of a turbulent primary gas flow is a subject of interest in nuclear and combustion engineering, atmospheric dynamics and particle technology, amongst other areas. One of the characteristics of such flows is the high density ratio between the particles and the carrier flow. Because the primary flows in these cases are turbulent, the particle trajectories are random and the degree of randomness influences the rate of spread of the particles. It is of interest in each application area to be able to estimate the rate of spread of particles under the influence of the primary flow turbulence. The rate of spread can be characterised in terms of a 'dispersion coefficient'. Other, related, dispersion characteristics, such as the 'energy' of the particle fluctuations, may also be of interest.

Real turbulent flows are extremely complex in that turbulence characteristics are different at different points in the flow (inhomogeneity), or in different co-ordinate directions (anisotropy), or at different times (non-stationarity). Analysis of such flows generally requires numerical solution methods. However, a great deal of understanding can be gained by the study of much simpler (idealized) turbulent flows, which are assumed to be homogeneous, isotropic and stationary (see Reeks 1977, for example).

Tchen (1947) (see also Hinze 1975) developed a theory for such flows which showed long-time particle dispersion coefficients to be independent of particle inertia. Hinze (1975) also investigated the temporal development of the dispersion coefficient. This approach has been to some extent superseded by subsequent analyses (e.g. Reeks 1977; Mei *et al.* 1991) which require less drastic simplifying assumptions. However, the present paper returns to this approach because of its relative simplicity and also because the main interest here is in *development times* of the dispersion characteristics. It is shown here that, particularly for high-inertia particles, these development times are largely independent of the fluid flow behaviour. Given that the assumption about the fluid flow behaviour is the principal difference between Tchen's (1947) model and the later analyses, this result may be taken as sufficient justification for adopting the simpler approach.

It should be expected that the development of the particle dispersion characteristics will be dependent on the initial state of the particles. In the present paper, dispersion characteristics of particles released from a point source into a simple turbulent flow are investigated. Two different cases are studied. The first case is one where the particles are initially static and must acquire turbulence from interaction with the primary flow. The second case corresponds to a steady-state, in which the particle motion is fully turbulent. In this case it is assumed that, prior to release, the particles have been resident in the turbulent gas flow long enough to develop turbulence.

By making simplifying assumptions about the equation of motion satisfied by a discrete particle, general expressions for the dispersion characteristics are developed for both of the above cases. In section 2, expressions are developed for particle velocity auto-correlations, 'energy', integral time scale, dispersion coefficient etc. for both of the cases noted above.

These quantities all depend on the behaviour of the fluid velocity seen by a particle on its path through the fluid. By assuming a particular form for the auto-correlation of this velocity, analytical expressions can be developed for the particle dispersion characteristics. The integrals developed in section 2 are evaluated in section 3, and expressions are developed for the dispersion characteristics.

In section 4, the development of the solutions as functions of time is investigated. Development times for high-inertia particles are also developed as functions of the particle relaxation time, and it is shown in an appendix that these development times are in fact independent of the fluid velocity auto-correlation.

The analysis leads to expressions for the particle dispersion characteristics as functions of time. In addition to estimating development times for acquisition of turbulence, the results presented can also be used in the calibration of Lagrangian models for particle dispersion. For many models which take into account the 'crossing trajectories', 'inertia' and 'continuity' effects, the performance of the model for finite-inertia particles with finite drift velocity is not necessarily known *a priori*. Full understanding of such models therefore requires that the models be calibrated over a range of conditions. This leads to the question of how long simulations should be run in order to determine long-time dispersion statistics. Estimates are given in this paper of the development times for the particle dispersion coefficient. It is shown that a pseudo-dispersion coefficient can be defined, whose long-time value is equal to that of the true dispersion coefficient, but which develops much more quickly than the true dispersion coefficient. The pseudo-dispersion coefficient could therefore be of use to reduce computation times in the calibration process. It is also shown that a common method for approximating the dispersion coefficient (called here the effective dispersion coefficient) develops much *less* quickly than the true dispersion coefficient.

2. ANALYSIS

As noted in the introduction above, we are interested in gas/particle flows, for which the particle density ρ_p is much greater than that of the primary flow density ρ_f. In order to facilitate analytical treatment, further simplifications are required. It is assumed that the particle Reynolds number is small, so that the drag on a particle is determined by Stokes' law. In addition, we assume that particles are small compared with length scales characteristic of the fluid motions and no lift forces act on a particle.

Under these assumptions, the (one-dimensional) equation of the motion of a spherical particle in a turbulent fluid flow is given by

$$\frac{du_p}{dt} = \beta(u_f^p - u_p). \qquad [2.1]$$

In [2.1], u_p and u_f^p are the velocity of the particle and the fluid in the vicinity of the particle, respectively, and β is the reciprocal relaxation time of the particle, determined in the case of Stokesian drag by

$$\beta = \frac{18\mu_f}{\rho_p d_p^2}, \qquad [2.2]$$

where μ_f is the viscosity of the fluid and d_p is the diameter of the solid particle.

Equation [2.1] represents a considerable simplification of the equations of motion developed by Maxey & Riley (1983), Mei (1994), for example, Much of the simplification is due to the high density ratio, as a result of which the 'Basset' and 'added-mass' terms normally present in the equation of motion can be ignored. Even with this simplification, however, the particle drag may be nonlinear in the gas/particle relative velocity and analytical treatment such as that developed here is generally not possible. However, it is worthwhile noting that, for the dispersion characteristics in which we are interested here, "the dominant contribution is from the particles' response to large scales of fluid motion, for which the linear drag is asymptotically correct" (Reeks 1977). The assumption of linear Stokesian drag is therefore reasonably realistic.

When β is constant, [2.1] can be integrated to give

$$u_p(t) = u_p(0) + \beta\, e^{-\beta t} \int_0^t e^{\beta t'} u_f^p(t')\, dt'. \qquad [2.3]$$

In a turbulent flow, the fluid motion is random so that $u_f^p(t')$ is not known as a function of time, and therefore particle velocities are also unknown. However, given information regarding the statistical behaviour of the fluid flow, certain statistical characteristics of the particle motion can be determined. It is shown below that these characteristics are dependent on the state of the particles as they are released into the turbulent flow. Two initial states are considered here, the first in which the particles are initially static and must acquire turbulence from the turbulent fluid flow, and the second in which particles are assumed to have been influenced by the turbulent flow for a length of time sufficient that the particles' random motion is statistically stationary.

2.1. *Analysis of particle motion starting from rest*

Much of the random behaviour of the particle motions can be characterised in terms of the particle velocity auto-correlation, defined as

$$H'(t, \tau) = \langle u_p(t) u_p(t + \tau) \rangle, \qquad [2.4]$$

where the angled brackets indicate ensemble averaging. Forming the product $u_p(t)u_p(t + \tau)$ using [2.3], rearranging and taking the ensemble average,

$$H'(t, \tau) = \beta^2\, e^{-\beta(2t+\tau)} \int_0^t e^{\beta\theta} \int_{-\theta}^{t+\tau-\theta} e^{\beta(\theta+\zeta)} \langle u_f^p(\theta) u_f^p(\theta + \zeta) \rangle\, d\zeta\, d\theta. \qquad [2.5]$$

Knowledge of $H'(t, \tau)$ enables determination of the particle energy $u_p'^2$ as a function of time. Particle energy is defined here as

$$u_p'^2(t) = H'(t, 0), \qquad [2.6]$$

where it can be assumed without loss of generality that the mean particle velocity is zero. The particle energy is simply the variance of the random particle velocities at time t and is a measure of the intensity of the fluctuations of the particle velocity at this time. The normalised form for $H'(t, \tau)$ is then given by

$$R_p'(t, \tau) = \frac{H'(t, \tau)}{u_p'^2(t)}. \qquad [2.7]$$

$R_p'(t, \tau)$ represents a correlation coefficient relating particle velocities at time t to those at the later time $t + \tau$. At $\tau = 0$, the particle velocities are, of course, perfectly correlated and $R_p'(t, \tau) = 1$. As τ increases, $R_p'(t, \tau)$ decreases for a fixed value of t until eventually, as $\tau \to \infty$, $R_p'(t, \tau) \to 0$. The integral time scale of the particle motions is defined as

$$\tau_p = \int_0^\infty R_p'(t, \tau)\, d\tau, \qquad [2.8]$$

which can be interpreted as an 'average' time for which the particle velocities remain correlated.

The position of a particle situated initially at the origin is given by

$$X_p(t) = \int_0^t u_p(\theta)\,d\theta. \qquad [2.9]$$

Squaring [2.9], using [2.3], rearranging and taking the ensemble average, assuming that the particles are initially static, the mean-square of the particle displacement at time t is given by

$$\langle X_p^2(t)\rangle = u'^2\beta^2 \int_0^t \int_0^t \int_0^\theta \int_{-x}^{\eta - x} e^{-\beta(\theta - 2x)}\, e^{-\beta(\eta - z)} \langle u_f^p(x) u_f^p(x+z)\rangle\, dz\, dx\, d\eta\, d\theta. \qquad [2.10]$$

The particle dispersion coefficient is then defined as

$$D_p(t) = \frac{1}{2}\frac{d}{dt}\langle X_p^2(t)\rangle. \qquad [2.11]$$

The dispersion coefficient is a measure of the spreading rate of the particles at time t. The particle dispersion characteristics can be seen from [2.5] and [2.11] to depend on the auto-correlation of the fluid velocities following the particle

$$G'(t,\tau) = \langle u_f^p(t) u_f^p(t+\tau)\rangle. \qquad [2.12]$$

If it can be assumed that $G'(t,\tau)$ is independent of the starting time t, then $G'(t,\tau)$ is of the form

$$G'(t,\tau) = u'^2 R_f^p(\tau), \qquad [2.13]$$

where $R_f^p(\tau)$, which is a function only of τ, is the auto-correlation normalised so that $R_f^p(0) = 1$, and u' is the turbulence intensity. Here, $R_f^p(\tau)$ is a correlation coefficient relating the fluid velocity in the vicinity of a particle at time t to the fluid velocity around the same particle at the later time $t+\tau$ (and this correlation coefficient is furthermore assumed to be independent of the 'starting' time t). The integral time scale of the fluid velocities following a particle is defined as

$$\tau_f^p = \int_0^\infty R_f^p(\tau), \qquad [2.14]$$

and represents an average time during which a particle remains associated with a particular 'eddy' in the turbulent flow.

Using [2.12] and [2.13], and substituting into [2.5],

$$H'(t,\tau) = u'^2\beta^2\, e^{-\beta(2t+\tau)} \int_0^t e^{\beta\theta} \int_{-\theta}^{t+\tau-\theta} e^{\beta(\theta+\zeta)} R_f^p(\theta)\, d\zeta\, d\theta. \qquad [2.15]$$

Equation [2.10] can be written as

$$\langle X_p^2(t)\rangle = \beta^2 \int_0^t \int_0^t \int_0^\theta \int_{-x}^{\eta - x} e^{-\beta(\theta - 2x)}\, e^{-\beta(\eta - z)} u'^2 R_f^p(z)\, dz\, dx\, d\eta\, d\theta. \qquad [2.16]$$

It is clear from [2.6], [2.8], [2.11], [2.15] and [2.16] that, if the fluid velocity auto-correlation along a particle path, $R_f^p(\tau)$, is known, many of the dispersion characteristics of initially static non-fluid particles can be found.

2.2. Analysis of motions of initially-excited particles

Pismen & Nir (1978) show that the limiting form for $H'(t,\tau)$ as $t \to \infty$ is given by

$$H(\tau) = \lim_{t\to\infty}\langle u_p(t) u_p(t+\tau)\rangle = u'^2\frac{\beta}{2}\int_{-\infty}^{\infty} e^{-\beta|t-\tau|} R_f^p(t)\, dt. \qquad [2.17]$$

$R_f^p(t)$ can be assumed to be symmetrical in time. It follows from this equation that $H(\tau)$ is symmetric in τ. The steady-state particle energy is given by

$$u_p'^2 = H(0) = \beta \int_0^\infty e^{-\beta t} u'^2 R_f^p(t)\,dt. \qquad [2.18]$$

The normalised form for $H(\tau)$ is given by

$$R_p(\tau) = \frac{H(\tau)}{u_p'^2}. \qquad [2.19]$$

The integral time-scale of the particle motions is then defined as

$$\tau_p = \int_0^\infty R_p(\tau)\,d\tau. \qquad [2.20]$$

If the particles are situated initially at the origin but have random initial velocities with the same probability distribution as in the steady-state, then the statistical state of the system of particles and fluid velocities is equivalent to that found in the steady state. The particle velocity auto-correlation will therefore be given by [2.17]. The mean-squared displacement of particles situated initially at the origin is given in this case by

$$\langle X_p^2(t)\rangle = \left\langle \int_0^t \int_0^t u_p(\eta)u_p(\theta)\,d\eta\,d\theta \right\rangle. \qquad [2.21]$$

Rearranging this equation, making use of the symmetry of $H(\tau)$,

$$\langle X_p^2(t)\rangle = 2\int_0^t \int_0^{t'} H(\tau)\,d\tau\,dt', \qquad [2.22]$$

which is of the form derived by Taylor (1921) to predict the dispersion of fluid particles. The particle dispersion coefficient is given by [2.11].

In reality, $R_f^p(\tau)$ is dependent upon the particle inertia and 'drift' (Reeks 1977; Pismen & Nir 1978). Here, however, in the spirit of Tchen (1947) and Hinze (1975), it is assumed that $R_f^p(\tau)$ is independent of particle inertia or drift. The main purpose of the following analysis is to describe the temporal development of particle dispersion characteristics after release into a turbulent fluid flow. It is demonstrated in appendix A that development times for high inertia particles are independent of the fluid turbulence characteristics (i.e. the turbulence energy spectrum). It is reasonable to assume that development times for finite-inertia particles are only weakly dependent on details of the fluid turbulence. Although long-time values of the dispersion coefficient, particle energy etc. *are* dependent on the fluid turbulence, it is the development times for these long-time values that are of interest here. A simple linear form is assumed for the purpose of deriving closed-form solutions for particle dispersion characteristics. The linear form has been chosen since it approximates closely the forms of auto-correlation functions found in certain numerical models used to model particle dispersion (see Graham & James 1996; Graham 1996).

3. EXACT SOLUTIONS FOR PARTICLE DISPERSION CHARACTERISTICS

The chosen form of the fluid velocity auto-correlation function is given by

$$R_f^p(\tau) = \begin{cases} 1 - |\tau|/T, & \text{if } |\tau| \leq T; \\ 0, & \text{otherwise.} \end{cases} \qquad [3.1]$$

This form may be considered as a very simple approximation to auto-correlation functions measured in homogeneous turbulence. This form for the auto-correlation function arises naturally in eddy interaction models (Graham & James 1996) which are used extensively to predict particle dispersion in turbulent flows. This form arises in such models in two different situations:

(i) dispersion of arbitrary-mass particles (when body forces can be neglected) in homogeneous, isotropic and stationary turbulence (HIST), in which case $T = T_e$ is the 'eddy lifetime',
(ii) gravitational settling of heavy particles in HIST, in which case $T = L_e/V_s$, where L_e is the eddy length and V_s is the settling velocity.

In each case, τ_f^p, the integral scale of the fluid motions seen by particles, is equal to $T/2$, so that the long-time particle dispersion coefficient is given by

$$\bar{D} = u'^2 T/2$$

for all particles.

3.1. Particles initially at rest

Substituting [3.1] into [2.15], it is found that the particle velocity auto-correlation function becomes

$$\begin{aligned} H'(t,\tau) = \frac{u'^2}{4\beta T}&[\mathrm{e}^{\beta(T-2t-\tau)} - \mathrm{e}^{\beta(\tau-T)} + 2\,\mathrm{e}^{-\beta t}\beta(\tau + t - T)]\mathrm{sign}(T-(t+\tau)) \\ &+ [\mathrm{e}^{\beta(T-2t-\tau)} - \mathrm{e}^{-\beta(\tau+T)} + 2\beta\,\mathrm{e}^{-\beta(t+\tau)}(t-T)]\mathrm{sign}(T-t) \\ &- [\mathrm{e}^{\beta(T-\tau)} - \mathrm{e}^{\beta(\tau-T)} + 2\beta[\tau - T]]\mathrm{sign}(T-\tau) \\ &+ \mathrm{e}^{\beta T}(\mathrm{e}^{-\beta\tau} - 2\,\mathrm{e}^{-\beta(2t+\tau)}) + \mathrm{e}^{-\beta(T+\tau)} + \mathrm{e}^{-\beta t}\beta[2\,\mathrm{e}^{-\beta\tau}(t-T) + t + \tau - T] \\ &+ \mathrm{e}^{-\beta(2t+\tau)}(4 + 4\beta T) - 4\,\mathrm{e}^{-\beta\tau} - 2\beta\tau + 2\beta T]. \end{aligned} \qquad [3.2]$$

The steady-state value is found as the limit as $t \to \infty$

$$H(\tau) = \begin{cases} \dfrac{u'^2}{2\beta T}[\mathrm{e}^{-\beta T}(\mathrm{e}^{-\beta\tau} + \mathrm{e}^{\beta\tau}) - 2\,\mathrm{e}^{-\beta\tau} + 2\beta T[1 - \tau/T]], & \text{if } \tau \leqslant T; \\ \dfrac{u'^2}{2\beta T}[\mathrm{e}^{-\beta\tau}(\mathrm{e}^{-\beta T} + \mathrm{e}^{\beta T} - 2)], & \text{otherwise.} \end{cases} \qquad [3.3]$$

This is consistent with the expression derived for $H(\tau)$ in Graham & James (1996). The particle energy is given by

$$u_p'^2(t) = H'(t,0) = \begin{cases} \dfrac{u'^2}{\beta T}[2\beta \mathrm{e}^{-\beta t}[t - T] \quad + \mathrm{e}^{-2\beta t}(1 + \beta T) + \beta T - 1], & \text{if } t \leqslant T; \\ \dfrac{u'^2}{\beta T}[\mathrm{e}^{-2\beta t}(1 + \beta T - \mathrm{e}^{\beta T}) + \mathrm{e}^{-\beta T} + \beta T - 1], & \text{otherwise.} \end{cases} \qquad [3.4]$$

The steady-state particle energy is

$$u_p'^2 = \lim_{t\to\infty} u_p'^2(t) = \frac{u'^2}{\beta T}(\mathrm{e}^{-\beta T} + \beta T - 1). \qquad [3.5]$$

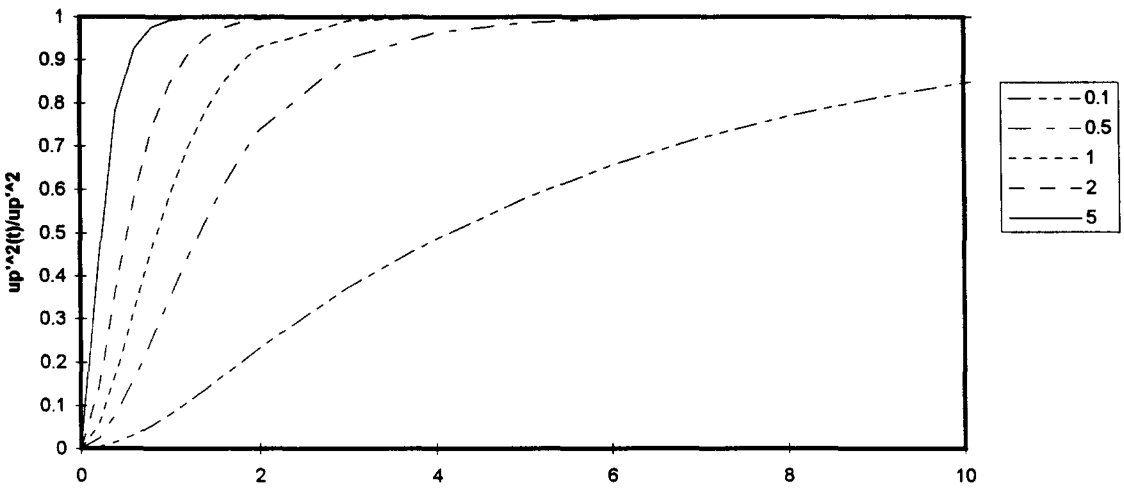
In the following discussion and on figures 1–7, non-dimensional time t^* is defined as $t/(T/2)$ and dimensionless reciprocal relaxation times β^* are given by $\beta T/2 = \beta\tau_f^p$. Asterisks are omitted for clarity of presentation, but all t and β should be understood to be dimensionless values. Other quantities are non-dimensionalised in an appropriate fashion. The values quoted in the legends of figures 1–7 represent values of β^*. The development of dispersion characteristics for particles with β^* in the range 0.1 (corresponding to high-inertia particles) to 5 (corresponding to tracer particles which follow the fluid turbulence fluctuations faithfully) is investigated.

The particle energy (normalised by the steady-state value given above) is plotted as a function of (dimensionless) time in figure 1(a), and as a function of βt in figure 1(b). Figure 1(a) shows that tracer particles (large β) have acquired most of their energy almost immediately after release (within

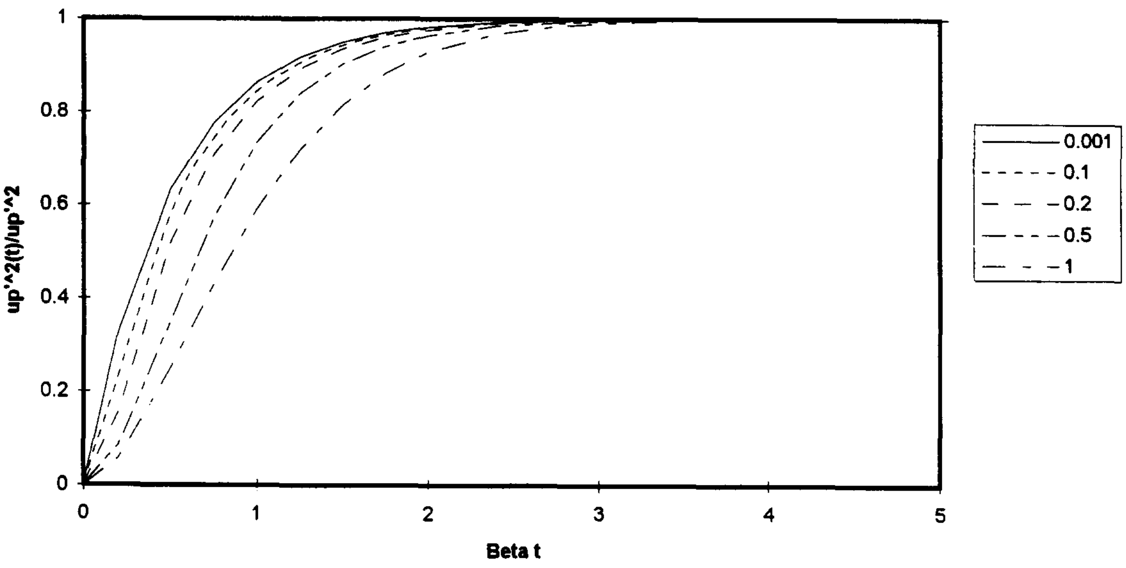
$t = 1$ for particles with $\beta > 5$). The relatively slow development of particle energy for high-inertia (low β) particles is also demonstrated. After $t = 10$ (i.e. 10 integral time scales), the energy of the particles with the greatest inertia ($\beta = 0.1$) is only around 80% of the long-time value. Figure 1(b) indicates that, when plotted as a function of βt, the development curves for sufficiently small β collapse onto a single limiting curve, corresponding to $\beta = 0$, and that development times are therefore dependent on the dimensionless group βt for high-inertia particles. The limiting case is discussed in section 4.2. below.

The numerator in [2.7] can be integrated to give a 'pseudo-dispersion' coefficient, defined by

$$D_{ps}(t) = \int_0^\infty H'(t, \tau)\, d\tau, \qquad [3.6]$$



(a)



(b)

Figure 1. (a) Time development of energy for initially-static particles; (b) energy of initially-static particles vs Beta t.

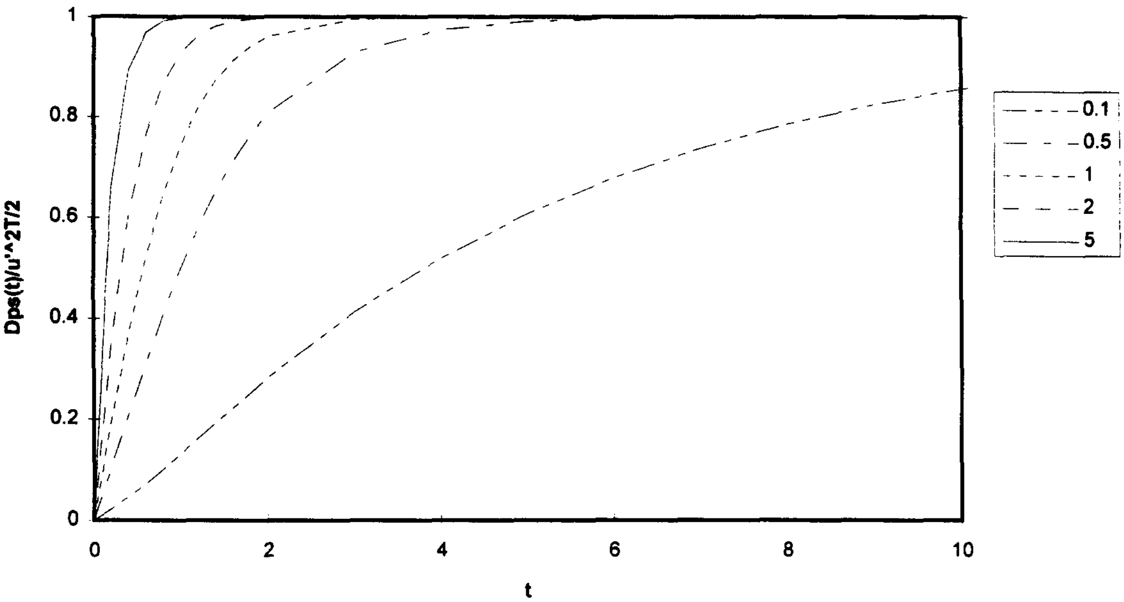


Figure 2. Time development of pseudo-dispersion coefficient for initially-static particles.

so that

$$D_{ps}(t) = \begin{cases} \dfrac{u'^2}{\beta^2 T}\left[-e^{-\beta t}\left[\dfrac{\beta^2(T-t)^2}{2} - e^{-2\beta t}(1+\beta T) + \beta(T-t) + 1\right] + \dfrac{\beta^2 T^2}{2}\right], & \text{if } t \leqslant T; \\ \dfrac{u'^2}{\beta^2 T}[e^{-2\beta t}(1+\beta T - e^{\beta T}) + \beta^2 T^2/2], & \text{otherwise.} \end{cases} \quad [3.7]$$

In the limit as $t \longrightarrow \infty$,

$$D_{ps}(t) \longrightarrow \frac{u'^2 T}{2}. \quad [3.8]$$

i.e. the value of the pseudo-dispersion coefficient approaches that of the dispersion coefficient. The pseudo-dispersion coefficient (normalised by $\bar{D}$) is plotted against time in figure 2. Development times are very similar to those of the particle energy.

The limiting integral time scale of the particle motions is given by use of [3.5] and [3.8],

$$\tau_p = \frac{u'^2 T/2}{(u'^2/(\beta T))(e^{-\beta T} + \beta T - 1)} = \frac{\beta T^2}{2(e^{-\beta T} + \beta T - 1)}. \quad [3.9]$$

For low-inertia particles ($\beta > 1$), the integral time scale approaches that of fluid particles (i.e. $T/2$). However, for particles with $\beta \ll 1$, the integral time scale of the particles is close to the particle 'relaxation time' $1/\beta$ (which represents the time required to reduce the particle velocity in a quiescent fluid by 63%).

Figure 3 shows the time-development of the particle integral time scale, normalised by its long-time value. Close to $t = 0$, the integral time scale is large for all particles. This is true even for the tracer-particles ($\beta = 5$) and is interpreted as signifying that, initially, the particle 'remembers' that it is stationary. For all β, however, the particle integral time scale is close to its long-time value within ten integral time scales, even for the highest-inertia particles ($\beta = 0.1$).

The mean-squared particle displacement is given by

$$\langle X_p^2(t)\rangle = \begin{cases} \dfrac{u'^2}{\beta^3 T}[e^{-\beta t}[\beta^2 t(2T-t) - 2\beta T - 4] + e^{-2\beta t}(\beta T + 1) \\ \quad - \beta^3 t^3/3 + \beta^2 t^2(\beta T + 1) - 2\beta t(\beta T + 1) + \beta T + 3], & \text{if } t \leqslant T; \\ \dfrac{u'^2}{\beta^3 T}[e^{-\beta t}[2\,e^{\beta T} + \beta^2 T^2 - 2\beta T - 2] + e^{-2\beta t}[1 + \beta T - e^{\beta T}] \\ \quad + \beta^3 T^2 t - \beta^3 T^3/3 - \beta^2 T^2 - \beta T + 1 - e^{-\beta T}], & \text{otherwise.} \end{cases} \qquad [3.10]$$

The particle dispersion coefficient is therefore given by

$$D_p(t) = \begin{cases} \dfrac{u'^2}{2\beta^2 T}[e^{-\beta t}[\beta^2 t^2 - 2t(\beta^2 T + \beta) + 4\beta T + 4] - 2e^{-2\beta t}(\beta T + 1) \\ \quad - \beta^2 t^2 + 2\beta t(\beta T + 1) - 2(\beta T + 1)], & \text{if } t \leqslant T; \\ \dfrac{u'^2}{2\beta^2 T}[e^{-\beta t}[-2\,e^{\beta T} - \beta^2 T^2 + 2\beta T + 2] \\ \quad - 2\,e^{-2\beta t}[1 + \beta T - e^{\beta T}] + \beta^2 T^2], & \text{otherwise.} \end{cases} \qquad [3.11]$$

Particle dispersion coefficients (normalised by the long-time value $\bar{D} = u'^2 T/2$) are plotted as a function of t in figure 4. Figure 4 shows that the development of the dispersion coefficient is much slower that development of particle energy and pseudo-dispersion coefficient, even for the tracer-particles. For example, the true dispersion coefficient for the $\beta = 0.1$ particles is only 35% of $\bar{D}$ at $t = 10$, compared with the 80% development attained at the same time by the pseudo-dispersion coefficient.

An 'effective' dispersion coefficient can be defined as

$$D_{\mathrm{eff}}(t) = \frac{1}{2t}\langle X_p^2(t)\rangle. \qquad [3.12]$$

As with the pseudo-dispersion coefficient, in the limit as $t \to \infty$, the effective dispersion coefficient is equal to the true long-time dispersion coefficient $\bar{D}$ and the effective dispersion

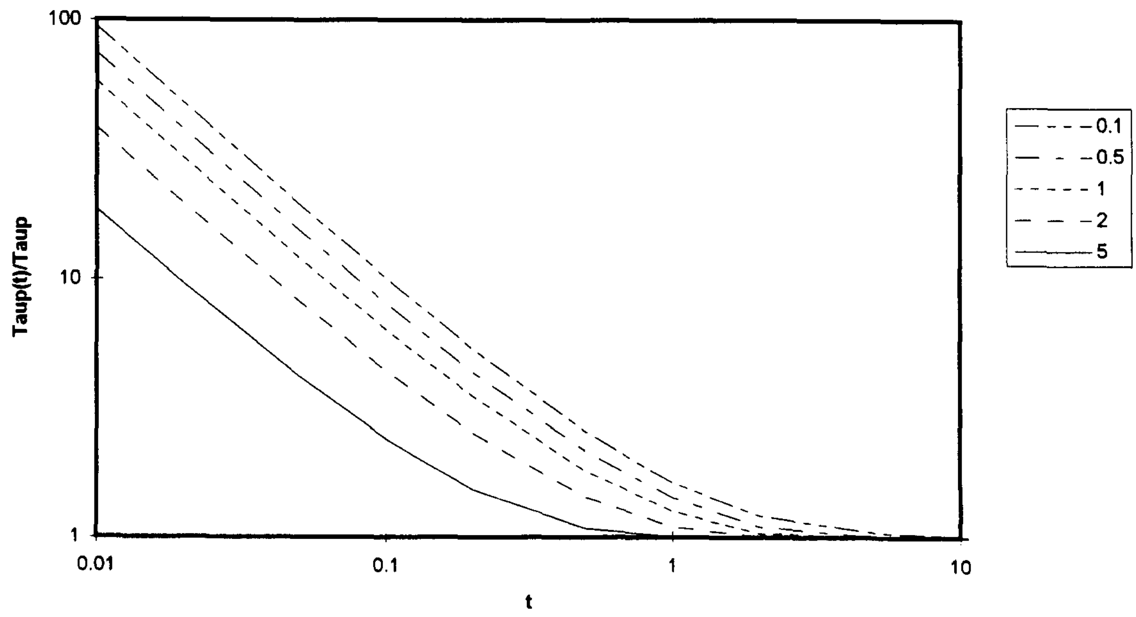


Figure 3. Time development of particle integral time scale for initially-static particles.

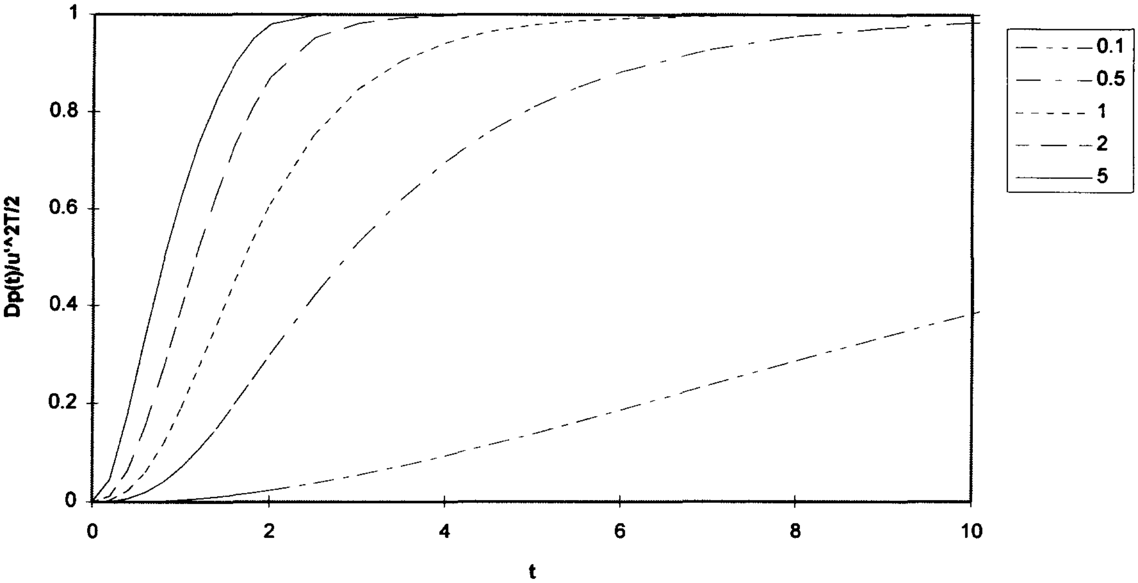


Figure 4. Time development of true dispersion coefficient for initially-static particles.

coefficient is often used to approximate the true dispersion coefficient in numerical simulations of particle dispersion. The expression for the effective dispersion coefficient can be determined by use of [3.10]. Figure 5 illustrates the effective dispersion coefficient (normalised by $\bar{D}$) plotted against t. The development of the effective dispersion coefficient is much slower than the true or pseudo-dispersion coefficients. Even for the tracer particles, the difference between D_{eff} and $\bar{D}$ is greater than 5% after $t = 20$. For the high-inertia particles, D_{eff} has developed to only 35% of its final value at this time. This is partially due to the slow acquisition of energy for the high-inertia particles, and partially due to the fact that [3.12] is a coarse approximation to the true dispersion

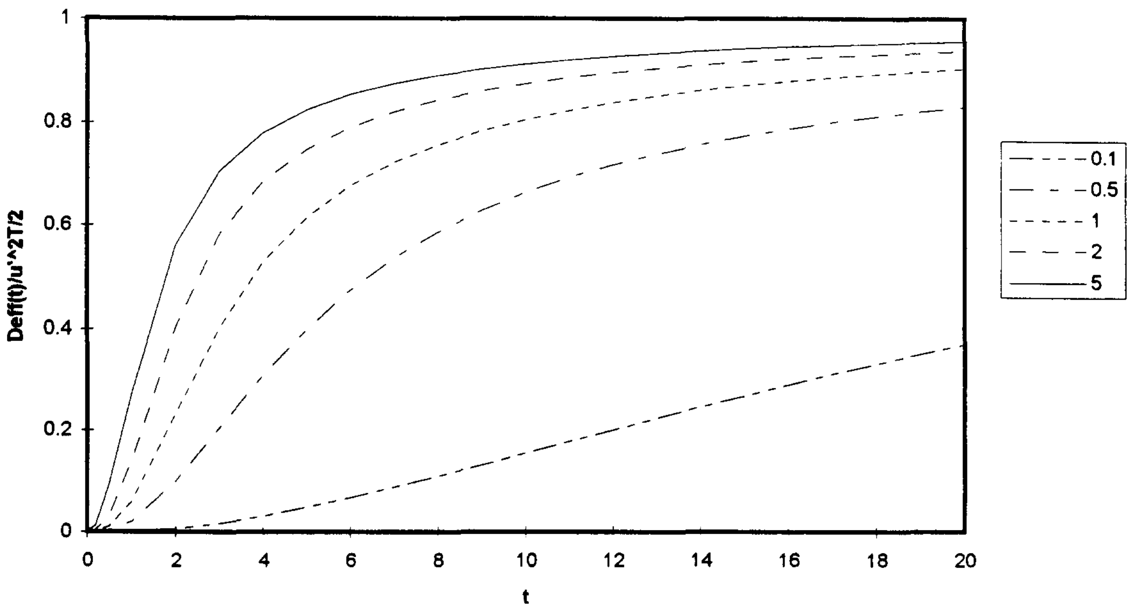


Figure 5. Time development of effective dispersion coefficient for initially-static particles.

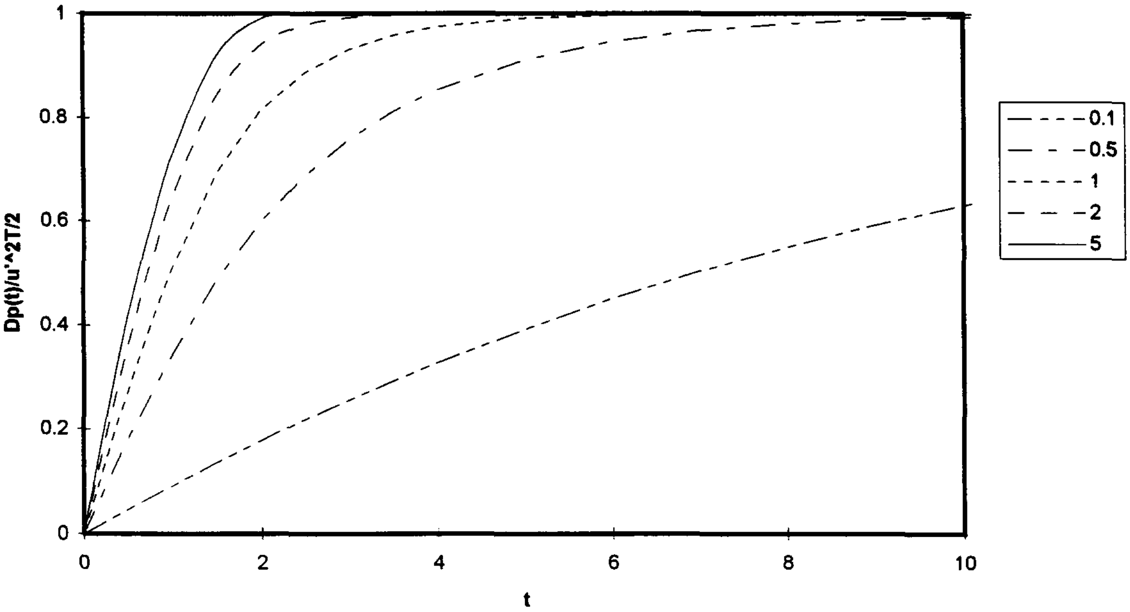


Figure 6. Time development of true dispersion coefficient for initially-excited particles.

coefficient, due to the initial non-linear development of the mean-squared particle displacement.

3.2. *Particles initially excited*

The long-time limiting form of $H(t, \tau)$ is given by [3.3]. The steady-state particle energy is given by use of [3.5]. The mean-squared particle displacement as a function of time is given by

$$\langle X_p^2(t)\rangle = \begin{cases} \dfrac{u'^2}{\beta^3 T}[e^{\beta t}\,e^{-\beta T} + e^{-\beta t}(e^{-\beta T} - 2) \\ \quad - \beta^3 t^3/3 + \beta^3 t^2 T - 2\,e^{-\beta T} - 2\beta t + 2], \quad \text{if } t \leqslant T; \\ = \dfrac{u'^2}{\beta^3 T}[e^{-\beta t}[e^{\beta T} + e^{-\beta T} - 2] \\ \quad + \beta^3 T^2 t - \beta^3 T^3/3 - 2\beta T + 2 - 2\,e^{-\beta T}], \text{ otherwise.} \end{cases} \qquad [3.13]$$

The dispersion coefficient is given by

$$D_p(t) = \begin{cases} \dfrac{u'^2}{2\beta^2 T}[e^{\beta t}\,e^{-\beta T} + e^{-\beta t}[2 - e^{-\beta T}] - \beta^2 t^2 + 2\beta^2 T t - 2], \text{ if } t \leqslant T; \\ \dfrac{u'^2}{2\beta^2 T}[e^{-\beta t}[2 - (e^{\beta T} + e^{-\beta T})] + \beta^2 T^2], \quad \text{otherwise.} \end{cases} \qquad [3.14]$$

The normalised dispersion coefficient is plotted as a function of t in figure 6. In the initial stages, near $t = 0$, development of the dispersion is clearly more rapid for all particles than was the case for initially static particles. Subsequent development is similar to the initially static particles, so that, for $\beta > 0.5$, the true dispersion coefficients for initially-static and initially-excited particles are at a similar state of development at $t = 10$. For the higher-inertia particles, the initial stages contribute more to the slow development of the dispersion coefficient, which is only 60% developed at $t = 10$.

An effective dispersion coefficient can again be defined as

$$D_{\text{eff}}(t) = \frac{1}{2t}\langle X_p^2(t)\rangle. \qquad [3.15]$$

The effective dispersion coefficient (plotted against t in figure 7) once more develops much more slowly than the true dispersion coefficient, as can be seen by comparison of figures 6 and 7. The development is, however, more rapid than was the case for the initially static particles, because of the time required for the initially static particles to acquire energy.

Time development of the various solutions is considered more fully in the next section. Having noted that all of the dispersion curves appear to have some limiting behaviour for high-inertia particles, solutions are expanded as Taylor series in the reciprocal particle time scale β, and development times are investigated as functions of βt for this case.

4. DEVELOPMENT TIMES FOR HEAVY PARTICLES

Inspection of [3.4] shows that in the limit as the initial sampling time t tends to infinity, the particle energy tends towards the steady-state value given by [3.5], so that the particles initially at rest do eventually attain the same energy as those released from the start. However, the time required to reach, say, 99% of the long-time value of the steady-state energy is evidently dependent upon β, the reciprocal relaxation time of the particle.

For fluid particles, $\beta \to \infty$ and the expressions for particle energy, mean-squared displacement and dispersion coefficient given by [3.4], [3.10] and [3.11] for the initially-static particles differ from the corresponding expressions [3.5], [3.13] and [3.14] for the initially-excited particles only near $t = 0$. The fluid particles are in the steady-state immediately after they are released from rest. However, non-fluid particles, for which β is finite, require some time to develop to their long-term steady state.

Of particular interest are the development times of heavy particles, for which β is small. The analysis carried out below involves expanding the expressions for particle characteristics as Taylor series about $\beta = 0$. Development times, where the solutions are developed close to their steady-state values, are then investigated.

4.1. *Particles initially at rest*

The characteristic functions for very heavy ($\beta \to 0$) particles are given by

$$u_p'^2(t) \sim \frac{u'^2\beta T}{2}(1 - e^{-2\beta t}), \quad \text{if } t \gg T, \qquad [4.1]$$

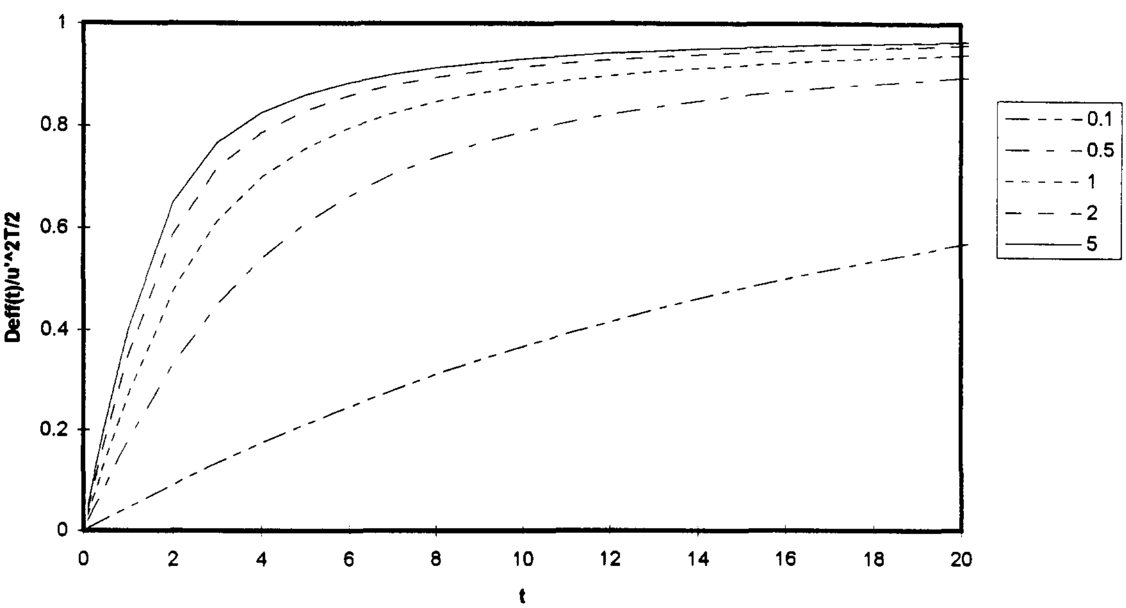


Figure 7. Time development for effective dispersion coefficient for initially-excited particles.

Table 1.

Quantity	Limit value	90% T_{dev}	95% T_{dev}	99% T_{dev}	α% T_{dev}
$u_p^2(t)$	$u'^2\beta T/2$	$1.15/\beta$	$1.50/\beta$	$2.30/\beta$	$\ln\left(\frac{1}{(1-\alpha/100)}\right)/2\beta$
$D_p(t)$	$u'^2T/2$	$2.97/\beta$	$3.68/\beta$	$5.30/\beta$	$\ln\left(\frac{1}{(1-\sqrt{(\alpha/100)})}\right)/\beta$
$D_{ps}(t)$	$u'^2T/2$	$1.15/\beta$	$1.50/\beta$	$2.30/\beta$	$\ln\left(\frac{1}{(1-\alpha/100)}\right)/2\beta$
$D_{eff}(t)$	$u'^2T/2$	$15/\beta$	$75/\beta$	$150/\beta$	$\frac{1.5}{(1-\alpha/100)\beta}$

and

$$D_{ps}(t) \sim \frac{u'^2T}{2}(1 - e^{-2\beta t}), \quad \text{if } t \gg T, \qquad [4.2]$$

so that

$$\tau_p(t) \sim \frac{1}{\beta}, \quad \text{if } t \gg T. \qquad [4.3]$$

The dispersion coefficient is given by

$$D_p(t) \sim u'^2T[\tfrac{1}{2}(1 + e^{-2\beta t}) - e^{-\beta t}], \quad \text{if } t \gg T. \qquad [4.4]$$

Using [3.13], the expression for the effective dispersion coefficient is

$$D_{eff}(t) \sim \frac{u'^2T}{2}\left[\frac{1}{\beta t}\left[2\,e^{-\beta t} - \frac{1}{2}e^{-2\beta t} - \frac{3}{2}\right] + 1\right], \quad \text{if } t \gg T. \qquad [4.5]$$

Table 1 lists the development times T_{dev} for the particle energy, and for the true, pseudo- and effective dispersion coefficients, where the development time is defined as the time beyond which the solution differs by less than $(1 - \alpha)$ from the steady-state solution. The general case in the right-hand column indicates the development times for arbitrary α close to 100%. It is shown in appendix A that the particle velocity auto-correlation function develops in a manner similar to the particle energy. Because the particle energy and the pseudo-dispersion coefficient are dependent on the development of this auto-correlation function (see section 2), development times are expected to be similar for these quantities. The true particle dispersion coefficient develops at only half of the rate of the pseudo-dispersion coefficient. Development of the effective dispersion coefficient is 50 times slower than the particle energy and 20 times slower than the true dispersion coefficient (values based on 95% development times).

4.2. Particles initially excited

The particle energy and integral time scale are always equal to their long-time limiting values. The development of the true particle dispersion coefficient for the initially-excited particles is given by

$$D_p(t) \sim \frac{u'^2T}{2}[1 - e^{-\beta t}], \quad \text{if } t \gg T. \qquad [4.6]$$

The development of the effective dispersion coefficient is characterised by

$$D_{eff}(t) \sim \frac{u'^2T}{2}\left[\frac{1}{\beta t}(e^{-\beta t} - 1) + 1\right], \quad \text{if } t \gg T. \qquad [4.7]$$

The development times for the dispersion coefficient and for the effective dispersion coefficient are given in table 2. Again, the development of the effective dispersion coefficient is much slower

Table 2.

Quantity	Limit value	$90\% T_{dev}$	$95\% T_{dev}$	$99\% T_{dev}$	$\alpha\% T_{dev}$
$D_p(t)$	$u'^2T/2$	$2.30/\beta$	$3.00/\beta$	$4.61/\beta$	$\ln\left(\frac{1}{(1-\alpha/100)}\right)/\beta$
$D_{eff}(t)$	$u'^2T/2$	$10/\beta$	$50/\beta$	$100/\beta$	$\frac{1}{(1-\alpha/100)\beta}$

(here by a factor of over 16 for 95% development) than the true dispersion coefficient. Comparison of the results for initially-static and initially-excited particles leads to the expected result that development times for the latter case are substantially less than for the former case. The development time for the effective dispersion coefficient for the initially-static particles is always approximately $1\frac{1}{2}$ times that for the initially-excited particles, irrespective of the level of development. However, it can be shown that, as the development level approaches 100%, the corresponding difference in development times for the dispersion coefficient is only approximately $\ln(2)/\beta$.

It is clear from the results that particle energy develops much more quickly than particle dispersion. In order to be fully-dispersive, one must expect that the particle energy be close to its long-time value. One might expect that the development time for the dispersion coefficient for initially-static particles would exceed the corresponding development time for the initially-excited particles by an amount equal to the development time of particle energy for the initially-static particles. Because increasing energy and dispersivity occur simultaneously, however, the development time for initially-static particles is less than this sum and is only around 25% greater than for initially-excited particles (95% development times).

The analysis shows that numerical particle dispersion methods in which particle dispersion characteristics of high-inertia particles are computed can be made considerably less time-consuming by the judicious choice of the method used to approximate long-time dispersion coefficients. Use of a pseudo-dispersion coefficient (formed by integration of the particle velocity auto-correlation) leads to a value close to the long-time value much more quickly than the real dispersion coefficient (formed by differentiation of the curve of mean-squared particle displacement vs time). The use of the effective dispersion coefficient requires extremely long computation times, even for initially excited particles, and should probably be avoided whenever possible.

5. CONCLUSIONS

This paper reports the results of an analytical study of the temporal development of dispersion characteristics (energy, velocity auto-correlations, dispersion coefficients, etc.) of particles released into a homogeneous, isotropic, stationary turbulent flow. The influence of two different sets of initial conditions for the particles on these characteristics has been investigated. The first case concerns particles initially at rest on release into the turbulent flow. The second case assumes that the particles are released so that the initial energy of the particles (i.e. the variance of the initial particle velocity distribution) is equal to the steady-state energy, where particles have been resident in the turbulent flow for a long time. The analysis assumes that the auto-correlation of the fluid velocities following a particle path has a simple linear form. The following conclusions can be formed:

(i) as expected, the development times for initially-static particles exceed the corresponding times for the initially-excited particles,
(ii) development times for high-inertia particles are independent of fluid turbulence characteristics,
(iii) particle energy and the pseudo-dispersion coefficient (both of which are derived from the particle velocity auto-correlation function) develop for the initially-static particles significantly more quickly than the true dispersion coefficient,
(iv) if long-time values of the particle dispersion coefficient are required for high-inertia particles, the true dispersion coefficient determined by differentiation of the mean-squared

displacement curve, or the pseudo-dispersion coefficient formed by integration of the particle velocity auto-correlation function should be used, with the latter method leading to reliable results more rapidly.

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APPENDIX A

Dispersion Characteristics for Heavy Particles—Arbitrary Energy Spectrum

The analysis presented in this paper has determined development times for particle motions subject to a particular (linear) form of the fluid velocity auto-correlation along the path of a particle, $R_f^p(t)$. It is shown below that the expressions for the particle velocity auto-correlation (and therefore particle energy and integral time-scale) and for both the effective and actual dispersion coefficients are independent of $R_f^p(t)$, and are therefore valid for any form of homogeneous, isotropic and stationary turbulence.

Initially-excited particles

The steady-state value of the particle velocity auto-correlation is given by

$$H(\tau) = \frac{u'^2\beta}{2}\int_{-\infty}^{\infty} e^{-\beta|t-\tau|}R_f^p(t)\,dt$$

$$= \frac{u'^2\beta}{2}\left(e^{-\beta\tau}\int_{-\infty}^{0} e^{\beta t}R_f^p(t)\,dt + e^{-\beta\tau}\int_{0}^{\tau} e^{\beta t}R_f^p(t)\,dt + e^{\beta\tau}\int_{\tau}^{\infty} e^{-\beta t}R_f^p(t)\,dt\right). \quad \text{[A1]}$$

Assuming that the integral time scale of the fluid motions $\tau_f^p = \int_0^\infty R_f^p(t)\,dt$ is small compared with $1/\beta$ and τ, the contribution to the integral from τ to ∞ is small and

$$H(\tau) = u'^2\beta\tau_f^p\,e^{-\beta\tau}. \quad \text{[A2]}$$

Consequently,

$$u_p'^2 = H(0) = u'^2\beta\tau_f^p \quad \text{[A3]}$$

and

$$R_p(\tau) = e^{-\beta\tau}, \qquad \text{[A4]}$$

so that

$$D_p(t) = u'^2 \int_0^t H(\tau) = u'^2\tau_f^p(1 - e^{-\beta t}), \qquad \text{[A5]}$$

$$\langle X_p^2(t)\rangle = 2\int_0^t \int_0^{t'} H(\tau)\,d\tau\,dt' = 2u'^2\tau_f^p\left(t + \frac{1}{\beta}(e^{-\beta t} - 1)\right) \qquad \text{[A6]}$$

and

$$D_{eff}(t) = \frac{\langle X_p^2(t)\rangle}{2t} = u'^2\tau_f^p\left(1 + \frac{1}{\beta t}(e^{-\beta t} - 1)\right). \qquad \text{[A7]}$$

Taking the Fourier transform of $H(\tau)$, and using the convolution theorem,

$$H(\tau) = \frac{\beta^2}{2\pi}\int_0^\infty \frac{E(\omega)\cos(\omega\tau)}{(\beta^2 + \omega^2)}\,d\omega \qquad \text{[A8]}$$

where

$$E(\omega) = 4u'^2 \int_0^\infty R_f^p(\tau)\cos(\omega\tau)\,d\tau \qquad \text{[A9]}$$

is a spectral energy function, so that

$$R_f^p(\tau) = \frac{1}{2\pi u'^2}\int_0^\infty E(\omega)\cos(\omega\tau)\,d\omega. \qquad \text{[A10]}$$

Initially-static particles

The particle velocity auto-correlation is given by

$$H'(t, \tau) = u'^2\beta^2 e^{-\beta(2t+\tau)} \int_0^t e^{\beta\theta} \int_{-\theta}^{t+\tau-\theta} e^{\beta(\theta+\zeta)} R_f^p(\theta)\,d\zeta\,d\theta$$

$$= \beta^2 e^{-\beta(2t+\tau)} \int_0^t e^{\beta\theta} \int_{-\theta}^{t+\tau-\theta} e^{\beta(\theta+\zeta)} \frac{1}{2\pi}\int_0^\infty E(\omega)\cos(\omega\zeta)\,d\omega\,d\zeta\,d\theta$$

$$= \frac{\beta^2}{2\pi}\int_0^\infty \frac{E(\omega)}{(\beta^2 + \omega^2)}[\cos(\omega t) - \cos(\omega(t + \tau))e^{-\beta\tau}$$

$$- \cos(\omega t)\,e^{-\beta(t+\tau)} + e^{-\beta(2t+\tau)}]\,d\omega. \qquad \text{[A11]}$$

Now

$$H(\tau) = u_p'^2\,e^{-\beta\tau} = \frac{\beta^2}{2\pi}\int_0^\infty \frac{E(\omega)\cos(\omega\tau)}{(\beta^2 + \omega^2)}\,d\omega \qquad \text{[A12]}$$

so that

$$\frac{\beta^2}{2\pi}\int_0^\infty \frac{E(\omega)\cos(\omega t)}{(\beta^2+\omega^2)}\,d\omega = u_p'^2\,e^{-\beta t}, \qquad \text{[A13]}$$

and, after some rearrangement,

$$H'(t,\tau) = u_p'^2\,e^{-\beta\tau}(1-e^{-2\beta t}). \qquad \text{[A14]}$$

Consequently,

$$u_p'^2(t) = u_p'^2(1-e^{-2\beta t}),\ t \gg \tau_f^p \qquad \text{[A15]}$$

and

$$\tau_p(t) = \frac{\int_0^\infty H'(t,\tau)\,d\tau}{u_p'^2(t)} = \frac{1}{\beta},\ t \gg \tau_f^p. \qquad \text{[A16]}$$

The mean-squared particle displacement at time t is given by

$$\langle X_p^2(t)\rangle = u'^2\beta^2\int_0^t\int_0^t\int_0^\theta\int_{-x}^{\eta-x} e^{-\beta(\theta-2x)}e^{-\beta(\eta-z)}\langle u_f^p(x)u_f^p(x+z)\rangle\,dz\,dx\,d\eta\,d\theta$$

$$= \beta^2\int_0^t\int_0^t\int_0^\theta\int_{-x}^{\eta-x} e^{-\beta(\theta-2x)}e^{-\beta(\eta-z)}\frac{1}{2\pi}\int_0^\infty E(\omega)\cos(\omega z)\,d\omega\,dz\,dx\,d\eta\,d\theta. \qquad \text{[A17]}$$

After some detailed algebra, this can be evaluated as

$$\langle X_p^2(t)\rangle = \frac{1}{2\pi}\int_0^\infty E(\omega)\left[\frac{(1-e^{-\beta t})^2}{(\beta^2+\omega^2)} - \frac{4\beta\sin(\omega t)}{\omega}(1-e^{-\beta t}) + \frac{4\beta^2}{\omega^2}(1-\cos(\omega t))\right]d\omega. \qquad \text{[A18]}$$

It can be shown that this expression is equivalent to

$$\langle X_p^2(t)\rangle = u'^2\tau_f^p\left[2t - \frac{1}{\beta}(1-e^{-\beta t})(3-e^{-\beta t})\right], \qquad \text{[A19]}$$

leading to

$$D_p(t) = u'^2\tau_f^p(1-e^{-\beta t})^2, \qquad \text{[A20]}$$

and

$$D_{eff}(t) = u'^2\tau_f^p\left[1 - \frac{1}{2\beta t}(1-e^{-\beta t})(3-e^{-\beta t})\right]. \qquad \text{[A21]}$$

For the turbulence spectrum considered in section 3, $\tau_f^p = T/2$, so that [A5] and [A7] for the initially excited particles are identical to [4.6], [4.7], and [A15], [A16], [A20] and [A21] for the initially static particles are identical to [4.1], [4.3], [4.4] and [4.5].